

Open mapping theorem:-

Definition:- An operator T on a normed space X into a normed space Y is called open if TA is an open subset of Y whenever A is an open subset of X i.e. T maps open sets into open sets.

1. Statement:- Let B and B' be Banach spaces. If T is a continuous linear transformation of B onto B' , then T is an open mapping.

Proof:- We are given that the linear transformation $T: B \rightarrow B'$

is continuous and onto and we want to show that T is an open map, that is, that $T(U)$ is an open set in B' for every open set U in B .

Let $y \in T(U)$ be arbitrary.

Then $y = T(x)$ for some $x \in U$.

Since U is an open set in B , there exists an open sphere $S(x, r)$ in B centred at x such that $S(x, r) \subset U$.

But as remarked earlier we can write

$S(x, r) = x + S_r$, where S_r is an open sphere in B

centred at origin. Thus

$$x + S_r \subset U \quad (1)$$

By our lemma, there exists an open sphere s'_ϵ in B' centred at origin such that $s'_\epsilon \subset T(S_r)$.

$$\therefore y + s'_\epsilon \subset y + T(S_r) = T(x) + T(S_r) = T(x + S_r)$$

$$\text{or } s'_\epsilon(y, \epsilon) \subset T(x + S_r) \quad [\because y + s'_\epsilon = s'_\epsilon(y, \epsilon)]$$

$$\subset T(U) \text{ by (1)}$$

Thus we have shown that to each point $y \in T(U)$ there exists an open sphere in B' centred at y and contained in $T(U)$ and consequently $T(U)$ is an open set. ~~thus~~

Proved.

2. Theorem:- Let B and B' be Banach spaces and let T be a one-one continuous linear transformation

of B onto B' . Then T is a homomorphism. In particular, T^{-1} is automatically continuous.

Proof: — Since T is one-one onto and continuous, we need only prove that T is an open mapping. So the proof of this theorem is exactly the same as of the previous theorem.

Note that the hypothesis of theorem 1 is sufficient to infer that T is a homomorphism in as much as T is not given to be a one-one mapping there. It is this additional property which makes T a homomorphism.

Theorem 2 is after known as Banach's Theorem.

3 Theorem: Let P be a projection on a Banach space B and let M and N be its range and null space respectively. Then M and N are closed linear manifolds (subspaces) of B such that $B = M \oplus N$.

Proof: — P is a projection on a Banach space
 \Leftrightarrow (i) P is a projection in the algebraic sense and

(ii) P is continuous.

But (i) implies that $B = M \oplus N$ where M and N are range and null spaces of P respectively. Now we use (ii) to prove that M and N are closed subspaces of B .

By definition of null space,

$$N = \{x : P(x) = 0\} = P^{-1}(\{0\}).$$

Since P is continuous and $\{0\}$ is closed in B , it follows that $P^{-1}(\{0\})$ is a closed subspace of B .

$$\text{Again, } M = \{x : P(x) = x\} = \{x : (I-P)(x) = 0\}.$$

i.e. M is the null space of $I-P$. But, since the identity map I is always continuous, $I-P$ is also continuous.

Hence, as shown above, its null space M must be closed.

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